

DISTRIBUTION OF SEQUENCES GENERATED BY CERTAIN SIMPLY-CONSTRUCTED NORMAL NUMBERS

DEMI ALLEN AND SKY BREWER

ABSTRACT. In 1949 Wall showed that $x = 0.d_1d_2d_3\dots$ is normal if and only if $(0.d_nd_{n+1}d_{n+2}\dots)_n$ is a uniformly distributed sequence. In this article, we consider sequences which are slight variants on this. In particular, we show that certain normal numbers of the form $0.a_na_{n+1}a_{n+2}\dots$, where a_n is a sequence of positive integers, give rise in a rather natural way to sequences which are not uniformly distributed. Motivated by a result of Davenport and Erdős we also show that for a non-constant integer polynomial the sequence $(0.f(n)f(n+1)f(n+2)\dots)_n$ is not uniformly distributed.

1. INTRODUCTION AND STATEMENT OF RESULTS

A number, α , is said to be normal to the base b if the frequencies of strings of digits in the b -adic expansion are as would be expected if the digits were completely random. In his 1933 paper, [Cha33], Champernowne exhibited a selection of numbers normal to the base 10 with simple constructions. Most notable of these was the so-called Champernowne's number, namely the number $0.1234567891011121314\dots$ constructed by concatenating all of the natural numbers in (ascending) order after the decimal point - throughout we will denote this number by θ . In 1949 it was proved by Wall in his PhD thesis [Wal49] that a real number α is normal to the base b if and only if the sequence $(b^n\alpha)_n$ is uniformly distributed modulo 1. Thus we know that the sequence $(10^n\theta)_n$ is uniformly distributed modulo 1. A natural question therefore is: what about the sequence $(x_n)_n = 0.(n)(n+1)(n+2)(n+3)\dots$ where the n th term is essentially constructed by taking θ but starting from the natural number n after the decimal point? For example the 20th term of the sequence $\{10^n\theta\}$ would be $0.516171819\dots$ whereas the 20th term of the sequence which we are now concerned with is $x_{20} = 0.202122232425\dots$. We ask the following, is this sequence uniformly distributed modulo 1 over a suitable subinterval of the unit interval?

In this paper we answer this question, in the negative, and consider various related questions. In particular, Davenport and Erdős showed in [DE52] that: given a polynomial $p : \mathbb{N} \rightarrow \mathbb{N}$ of degree ≥ 1 the number $0.f(1)f(2)f(3)\dots$ is normal. We consider also the distribution of the sequence $x_n = 0.f(n)f(n+1)f(n+2)\dots$.

Before stating our first result we introduce some necessary terminology and notation which will be used throughout.

Date: October 30, 2015.

1991 Mathematics Subject Classification. Primary 11K06; Secondary 11K16, 11J71.

Key words and phrases. Normal numbers, Uniformly distributed sequences.

For positive real numbers x we will denote by $\lfloor x \rfloor$ the integer part of x and by $\{x\}$ the fractional part of x . For a sequence of real numbers $(x_n)_n$ and $E \subseteq [0, 1]$ we will denote by $A(E; N; (x_n)_n)$ the number of x_n satisfying both $\{x_n\} \in E$ and $1 \leq n \leq N$.

Returning to the sequence $(x_n)_n$ corresponding to Champernowne's number defined above we note that it has no values in the interval $[0, 0.1]$ but that it is dense in the interval $[0.1, 1)$. So we consider uniform distribution over such an interval using the following definition, which is based on Definition 1.1 given in [KN74, Chapter 1].

Definition 1.1. We will say that a sequence of real numbers $(x_n)_n$ is *uniformly distributed modulo 1 over* $[\alpha, \beta) \subseteq [0, 1)$, which we shall henceforth abbreviate to *u.d. mod 1 over* $[\alpha, \beta)$, if for any pair of real numbers $\alpha \leq a < b \leq \beta$ we have

$$(1.1) \quad \lim_{N \rightarrow \infty} \frac{A([a, b); N; (x_n)_n)}{N} = \frac{b - a}{\beta - \alpha}.$$

Our first result is inspired by the normality of Champernowne's number and Wall's result.

Theorem 1.2. *The sequence $(x_n)_n$ of real numbers defined by $x_n = 0.(n)(n+1)(n+2)(n+3)\dots$, where the n th term is formed by concatenating all of the natural numbers in order from n onwards, is not u.d. mod 1 over $[0.1, 1)$.*

One can prove this quite easily by observing that, for any natural number J , upon reaching the term x_{10^J} the next 10^J terms will begin with a 1 immediately after the decimal point. That is; for each $J \in \mathbb{N}$ at least half of the terms up to the term $x_{2 \times 10^J}$ begin with a first decimal digit 1. More precisely, for $J \in \mathbb{N}$;

$$\frac{A([0.1, 0.2); 2 \times 10^J; (x_n)_n)}{2 \times 10^J} = \frac{\#\{x_n \in [0.1, 0.2) : n \leq 2 \times 10^J\}}{2 \times 10^J} \geq \frac{1}{2}.$$

Comparing this with Definition 1.1 the result of Theorem 1.2 follows. The point is that there are too many terms of the sequence in the interval $[0.1, 0.2)$ infinitely often.

As well as proving the normality of θ in [Cha33] Champernowne also highlights a few other very natural constructions of decimals which turn out to be normal - for example the number $0.46891012141516182021\dots$ formed by concatenating all of the composite numbers in ascending order. The motivation for our next result is one such construction considered by Champernowne in [Cha33], namely:

Theorem (Champernowne). *If k is any positive number and a_n denotes the integral part of kn , then the decimal $0.a_1a_2\dots a_n\dots$ is normal in the scale of ten.*

We remark that Champernowne does not provide an explicit proof of this statement (or indeed of the normality of $0.46891012141516182021\dots$) in [Cha33]. However, this can be verified by Copeland and Erdős' result in [CE46]. So, taking $k \in \mathbb{N}$ in Champernowne's Theorem stated above we obtain the normal number $0.k(2k)(3k)(4k)\dots$. However, along the same lines as Theorem 1.2, when we ask the analogous question here to the one posed in the introduction regarding Champernowne's number, we obtain the following result.

Theorem 1.3. *Let $k \in \mathbb{N}$ be arbitrary. Then, the sequence $(x_n)_n$ defined by*

$$x_n = 0.(kn)(k(n+1))(k(n+2))(k(n+3)) \dots$$

is not u.d. mod 1 over $[0.1, 1)$.

Finally, motivated by the result of Davenport and Erdős in [DE52] we establish the following theorem.

Theorem 1.4. *Let $f(n) = c_d n^d + c_{d-1} n^{d-1} + \dots + c_1 n + c_0$ be a non-constant polynomial with real coefficients such that for $n \in \mathbb{N}$ we have $f(n) \in \mathbb{N}$. Define a sequence by $x_n = 0.f(n)f(n+1)f(n+2)f(n+3) \dots$. Then, the sequence $(x_n)_n$ is not u.d. mod 1 over $[0.1, 1)$.*

2. PROOFS

We begin this section with a lemma which is the key to establishing Theorem 1.3.

Lemma 2.1. *Let $a_n = kn$ and define the sequence $(x_n)_n$ by $x_n = 0.a_n a_{n+1} a_{n+2} \dots$ then*

$$A\left([0.1, 0.2); \left\lfloor \frac{2 \times 10^J}{k} \right\rfloor; (x_n)_n\right) = \sum_{i=0}^J \left(\left\lfloor \frac{10^i}{k} \right\rfloor + O(1) \right).$$

Proof of Lemma 2.1. We observe that $A\left([0.1, 0.2), \left\lfloor \frac{2 \times 10^J}{k} \right\rfloor; (x_n)_n\right)$ counts the number of terms a_n with leading digit 1 and $n \leq \left\lfloor \frac{2 \times 10^J}{k} \right\rfloor$. That is, the number of terms a_n satisfying $10^j \leq a_n < 2 \times 10^j$ for some $j \in \mathbb{N}$ with $0 \leq j \leq J$. So, we may write

$$\begin{aligned} A\left([0.1, 0.2), \left\lfloor \frac{2 \times 10^J}{k} \right\rfloor; (x_n)_n\right) &= \sum_{j=0}^J \#\{n : 10^j \leq a_n < 2 \times 10^j\} \\ &= \sum_{j=0}^J \left(\left\lfloor \frac{10^j}{k} \right\rfloor + O(1) \right), \end{aligned}$$

which is the desired result. \square

We may now proceed to prove Theorem 1.3.

Proof of Theorem 1.3. We begin by recalling the fact that, given a bounded sequence of real numbers $(x_n)_n$, we have

$$\overline{\lim}_{n \rightarrow \infty} x_n \geq \overline{\lim}_{k \rightarrow \infty} x_{n_k}$$

where $(x_{n_k})_{n_k}$ is any subsequence of x_n .

Using this fact in conjunction with Lemma 2.1 we will show that

$$\overline{\lim}_{N \rightarrow \infty} A([0.1, 0.2); N; (x_n)_n) \geq \frac{5}{9}.$$

This suffices to show that the sequence $(x_n)_n$ is not u.d. mod 1 over $[0.1, 1)$ since, if it were, we would have

$$\overline{\lim}_{N \rightarrow \infty} \frac{A([0.1, 0.2); N; (x_n)_n)}{N} = \lim_{N \rightarrow \infty} \frac{A([0.1, 0.2); N; (x_n)_n)}{N} = \frac{0.2 - 0.1}{1 - 0.1} = \frac{1}{9} < \frac{5}{9}.$$

We will consider the value of $\frac{A([0.1, 0.2]; N; (x_n)_n)}{N}$ evaluated at each of the points of the subsequence $(n_j)_j$ of the natural numbers defined by

$$n_j = \left\lfloor \frac{2 \times 10^j}{k} \right\rfloor \text{ for all } j > \frac{\log k - \log 2}{\log 10}$$

(the condition imposed on j ensures that $\left\lfloor \frac{2 \times 10^j}{k} \right\rfloor \geq 1$).

By Lemma 2.1 we have

$$A([0.1, 0.2]; n_j; (x_n)_n) = \sum_{i=0}^j \left(\left\lfloor \frac{10^i}{k} \right\rfloor + O(1) \right).$$

From which it follows that

$$\begin{aligned} \lim_{j \rightarrow \infty} \frac{A([0.1, 0.2]; n_j; (x_n)_n)}{n_j} &= \lim_{j \rightarrow \infty} \frac{\sum_{i=0}^j \left(\left\lfloor \frac{10^i}{k} \right\rfloor + O(1) \right)}{\left\lfloor \frac{2 \times 10^j}{k} \right\rfloor} \\ &\geq \lim_{j \rightarrow \infty} \frac{k \sum_{i=0}^j \frac{10^i}{k} + O(j)}{2 \cdot 10^j} \\ &= \frac{k}{2} \cdot \frac{1}{k} \cdot \sum_{i=0}^{\infty} \frac{1}{10^i} = \frac{5}{9}, \end{aligned}$$

where the last equality is obtained by observing that $\sum_{i=0}^{\infty} \frac{1}{10^i}$ is a geometric series.

Since we have now shown that there is a subsequence of the natural numbers $(n_j)_j$ for which

$$\lim_{j \rightarrow \infty} \frac{A([0.1, 0.2]; n_j; (x_n)_n)}{n_j} \geq \frac{5}{9}$$

it follows that

$$\lim_{N \rightarrow \infty} \frac{A([0.1, 0.2]; N; (x_n)_n)}{N} \geq \frac{5}{9}.$$

This concludes the proof of Theorem 1.3. \square

The main tool we use to prove Theorem 1.4 is a generalisation of Lemma 2.1.

Lemma 2.2. *Let $f(n) = c_d n^d + c_{d-1} n^{d-1} + \dots + c_1 n + c_0$ be a polynomial with real coefficients and of degree ≥ 1 such that for $n \in \mathbb{N}$ we have $f(n) \in \mathbb{N}$. Define a sequence by $x_n = 0.f(n)f(n+1)f(n+2)f(n+3)\dots$. Then for $J \in \mathbb{N}$ we have*

$$A([0.1, 0.2]; f^{-1}(2 \times 10^J); (x_n)_n) = \left(\frac{2^{1/d} - 1}{c_d^{1/d}} \right) \sum_{i=1}^J (10^{i/d}) + O(J).$$

In order to prove Lemma 2.2, and subsequently Theorem 1.4, we require the following observation:

Lemma 2.3. *Let*

$$(2.1) \quad f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

be a polynomial of degree d and let $g(m)$ be its eventually monotonically increasing inverse (i.e. $f(g(m)) = m$). Then $g(m) = m^{1/d} c_d^{-1/d} + \varepsilon(m)$ and as $m \rightarrow \infty$ we have $\varepsilon(m) = O(1)$.

Proof. First, we substitute $n = m^{1/d} c_d^{-1/d} + \varepsilon$ into (2.1) to get

$$f(m^{1/d} c_d^{-1/d} + \varepsilon) = c_d (m^{1/d} c_d^{-1/d} + \varepsilon)^d + c_{d-1} (m^{1/d} c_d^{-1/d} + \varepsilon)^{d-1} + \cdots + c_0.$$

Using a combination of the Binomial theorem and Taylor expansions we may establish that

$$\varepsilon = -\frac{c_{d-1}}{d c_d^{1-2/d}} + O(m^{-2/d}).$$

□

With this in mind, we may now proceed to prove Lemma 2.2.

Proof of Lemma 2.2. The idea behind this proof is the same as that used to establish Lemma 2.1.

Let $f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$ be a polynomial with real coefficients such that for $n \in \mathbb{N}$ we have $f(n) \in \mathbb{N}$. Define a sequence by $x_n = 0.f(n)f(n+1)f(n+2)f(n+3)\dots$. By Lemma 2.3 we have $f^{-1}(n) = c_d^{-1/d} n^{1/d} + O(1)$. Thus, since $f(n)$ is increasing, we observe that for $J \in \mathbb{N}$ we have

$$\begin{aligned} & A([0.1, 0.2]; f^{-1}(2 \times 10^J); (x_n)_n) \\ &= (f^{-1}(2 \times 10^J) - f^{-1}(10^J)) + (f^{-1}(2 \times 10^{J-1}) - f^{-1}(10^{J-1})) + \dots \\ & \quad + (f^{-1}(20) - f^{-1}(10)) + O(J) \\ &= \sum_{i=1}^J (f^{-1}(2 \times 10^i) - f^{-1}(10^i)) + O(J) \\ &= \sum_{i=1}^J \left(\left(\frac{2 \times 10^i}{c_d} \right)^{1/d} + O(1) - \left(\frac{10^i}{c_d} \right)^{1/d} - O(1) \right) + O(J) \\ &= \frac{1}{c_d^{1/d}} \sum_{i=1}^J ((2 \times 10^i)^{1/d} - (10^i)^{1/d}) + O(J) \\ &= \frac{(2^{1/d} - 1)}{c_d^{1/d}} \sum_{i=1}^J (10^{i/d}) + O(J), \end{aligned}$$

as required. □

The proof of Theorem 1.4 follows from Lemma 2.2 essentially as Theorem 1.3 follows from Lemma 2.1 as we shall now see.

Proof of Theorem 1.4. In a similar fashion to the proof of Theorem 1.3 we will show that

$$\overline{\lim}_{J \rightarrow \infty} \frac{A([0.1, 0.2]; f^{-1}(2 \times 10^J); (x_n)_n)}{f^{-1}(2 \times 10^J)} > \frac{1}{9}.$$

This would suffice to show that the sequence $(x_n)_n$ is not u.d. mod 1 over $[0.1, 1)$ since it would show that

$$\overline{\lim}_{N \rightarrow \infty} \frac{A([0.1, 0.2]; N; (x_n)_n)}{N} > \frac{1}{9}.$$

Now, by using Lemma 2.2 and the formula for an infinite geometric series, for $J \in \mathbb{N}$ we have

$$\begin{aligned} \overline{\lim}_{J \rightarrow \infty} \frac{A([0.1, 0.2]; f^{-1}(2 \times 10^J); (x_n)_n)}{f^{-1}(2 \times 10^J)} &= \overline{\lim}_{J \rightarrow \infty} \frac{\left(\frac{2^{1/d} - 1}{c_d^{1/d}} \right) \sum_{i=1}^J (10^{i/d}) + O(1)}{\left(\frac{2 \times 10^J}{c_d} \right)^{1/d} + O(1)} \\ &\geq \overline{\lim}_{J \rightarrow \infty} \frac{\left(\frac{2^{1/d} - 1}{c_d^{1/d}} \right) \sum_{i=1}^J (10^{i/d}) + O(1)}{2 \times \left(\frac{2 \times 10^J}{c_d} \right)^{1/d}} \\ &= \lim_{J \rightarrow \infty} \frac{\left(\frac{2^{1/d} - 1}{c_d^{1/d}} \right) \sum_{i=1}^J (10^{i/d})}{2 \times \left(\frac{2 \times 10^J}{c_d} \right)^{1/d}} \\ &= \frac{(2^{1/d} - 1)}{2 \times 2^{1/d}} \lim_{J \rightarrow \infty} \sum_{i=0}^{J-1} 10^{-\frac{i}{d}} \\ &= \frac{5^{1/d}(2^{1/d} - 1)}{2(10^{1/d} - 1)}. \end{aligned}$$

Next, we will show that for any $n \in \mathbb{N}$ we have

$$\frac{5^{1/n}(2^{1/n} - 1)}{2(10^{1/n} - 1)} \geq \frac{\log 2}{\log 10} > \frac{1}{9}.$$

To save on notation, let us define $y_n = \frac{5^{1/n}(2^{1/n} - 1)}{2(10^{1/n} - 1)}$. We observe that $(y_n)_n$ is a monotonically decreasing sequence. Furthermore, by considering $\frac{5^x(2^x - 1)}{2(10^x - 1)}$ one may use l'Hôpital's rule to show that $\lim_{x \rightarrow 0} \frac{5^x(2^x - 1)}{2(10^x - 1)} = \frac{\log 2}{2 \log 10}$ which, in turn, shows that $\lim_{n \rightarrow \infty} y_n = \frac{\log 2}{2 \log 10}$.

Since $(y_n)_n$ is a monotonically decreasing sequence it follows that $y_n \geq \frac{\log 2}{2 \log 10}$ for all $n \in \mathbb{N}$. Thus, for any $d \in \mathbb{N}$,

$$\overline{\lim}_{J \rightarrow \infty} \frac{A([0.1, 0.2]; f^{-1}(2 \times 10^J); (x_n)_n)}{f^{-1}(2 \times 10^J)} \geq \frac{5^{1/d}(2^{1/d} - 1)}{2(10^{1/d} - 1)} \geq \frac{\log 2}{2 \log 10} > \frac{1}{9},$$

as claimed.

The proof of Theorem 1.4 is thus complete. \square

2.1. A comment on Benford's Law. One may be thinking at this point that perhaps these results are a consequence of Benford's Law [Rai76] - after all, considering leading digits (specifically the abundance of ones as a leading digit) is the crux of the proofs of Theorems 1.2, 1.3 and 1.4. We shall conclude by taking a moment to discuss this.

For sequences, instead of Benford's Law itself, one considers the notion of (strong) Benford sequences. Rather conveniently for us, Cigler proposed a characterisation of Benford sequences in terms of uniform distribution modulo 1 given below (see, for example, [Dia77]). For further discussion of this topic we refer the readers to, for example, [BNS10], [Dia77] and [Rai76].

Proposition 2.4. *The sequence $(a_i)_i$ is a strong Benford sequence if and only if $(\log_{10} a_i)_i$ is uniformly distributed modulo 1.*

It follows immediately from this characterisation that the sequence of natural numbers is not a strong Benford sequence as $(\log_{10} n)_n$ is not u.d. mod 1 (one possible way to show this is by modifying the argument of Example 2.4 in [KN74, Chapter 1]). Also $\log_b(f(n))$ is not u.d. mod 1 for all bases b and polynomials $f : \mathbb{N} \rightarrow \mathbb{N}$. To see this first note that we can represent any polynomial as $f(n) = (n - \alpha_1)(n - \alpha_2) \dots (n - \alpha_d)$ where d is the degree of the polynomial and the α_i are roots. Hence $\log_b(f(n)) = \sum_{i=1}^d \log_b(n - \alpha_i) \sim d \log_b(n)$ and so is not u.d. mod 1.

Acknowledgements. The authors would like to thank Christopher Hughes, Victor Beresnevich and Sanju Velani for providing useful comments on the manuscript. They are especially grateful to Chris for the many improvements to earlier drafts he has suggested.

DA and SB are supported by EPSRC Doctoral Training Grants EP/M506680/1 and EP/L505122/1, respectively.

REFERENCES

- [BNS10] V. Balázš, K. Nagasaka, and O. Strauch, *Benford's law and distribution functions of sequences in $(0, 1)$* , Mathematical Notes **88** (2010), no. 3-4, 449–463.
- [CE46] A. H. Copeland and P. Erdős, *Note on normal numbers*, Bull. Amer. Math. Soc. **52** (1946), 857–860.
- [Cha33] D. G. Champernowne, *The Construction of Decimals Normal in the Scale of Ten*, J. London Math. Soc. **S1-8** (1933), no. 4, 254.
- [DE52] H. Davenport and P. Erdős, *Note on normal decimals*, Canadian J. Math. **4** (1952), 58–63.
- [Dia77] P. Diaconis, *The distribution of leading digits and uniform distribution mod 1*, Ann. Probability **5** (1977), no. 1, 72–81.
- [KN74] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley-Interscience [John Wiley & Sons], New York, 1974. Pure and Applied Mathematics.
- [Rai76] R. Raimi, *The first digit problem*, American Mathematical Monthly **83** (1976), no. 7, 521–538.
- [Wal49] D. D. Wall, *Normal numbers*, University of California, Berkeley, 1949.

(D. Allen) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF YORK, YORK, UK, YO10 5DD
E-mail address, D. Allen: dda505@york.ac.uk

(S. Brewer) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF YORK, YORK, UK, YO10 5DD
E-mail address, S. Brewer: jaco.brewer@gmail.com